POTENTIAL AND ACTUAL INFINITESIMALS IN MODELS OF CONTINUUM

PAOLO GIORDANO AND MIKHAIL G. KATZ

Abstract. Cauchy's use of infinitesimals in defining key concepts of analysis and geometry such as continuity and center of curvature has been re-examined recently by Błaszczyk et al., Borovik et al., Bråting, Katz & Katz, and others. Were Cauchy's infinitesimals potential or actual?

With the advent of modern mathematical theories of actual infinitesimals, one can formally compare the notion of potential and actual infinitesimal. Is there a theory of continuum that expresses the relationship between these two concepts? Are such theories sufficiently powerful for a Leibnizian law of continuity to hold? In non-standard analysis, such questions are only partially answered by the works of Robinson, Lakatos, Cleave, Cutland et al., and Laugwitz concerning Cauchy's notion of infinitesimal.

We seek to answer such questions in two ways. The first approach is a critical revision of the non-standard approach, keeping in mind a working dichotomy between these two types of infinitesimals and a Leibnizian law for continuous relations. The second approach leads to nilpotent infinitesimals and a Leibnizian law in intuitionistic logic.

Contents

Key words and phrases. Bernoullian continuum; non-Archimedean rings; potential infinitesimals; actual infinitesimals; invertible infinitesimals; nilpotent infinitesimals.

P. Giordano was supported by a L. Meitner FWF (Austria; grant M1247-N13).

M.G. Katz was supported by the Israel Science Foundation grant 1517/12.

1. Introduction

The distinction between potential and actual infinity is somewhat puzzling to a modern reader. Such a distinction has been a common theme in philosophical commentary on infinity starting with Aristotle or even earlier. Yet the distinction does not lend itself easily to formalisation in the commonly accepted foundation for mathematics in the context of Zermelo–Fraenkel set theory. We will investigate such a distinction in the context of Cauchy's foundational work on infinitesimals.

Cauchy defined an infinitesimal in terms of variable quantities. More precisely, he wrote that a variable quantity *becomes* an infinitesimal if its values become arbitrarily small. Cauchy specifically points out that the limit of such a quantity is 0. Thus, the primitive notion in Cauchy's approach in one of a variable quantity, and both infinitesimals and limits are defined in terms of it. The meaning of the term *"becomes"* is subject to controversy. Some speculations on this subject may be found in Borovik & Katz [3]. At any rate, what is clear is that Cauchy never considered sequences other than null sequences when he wished to generate an infinitesimal. The only explicit example he gives of an infinitesimal is a (mildly non-monotone) null sequence. Having chosen a base infinitesimal, Cauchy proceeds to define infinitesimals of higher orders in terms of the base infinitesimal. All infinitesimals so generated are therefore defined by null sequences.

2. Potential and actual infinitesimals

In this section we propose a formalisation of the distinction between potential and actual infinitesimals. A general formal approach to actual infinitesimals is given by the following definition.

Definition 1. Let $(B, +, \cdot, \leq)$ be a preordered (unital) ring[1] and let $a \in B$. We say that a is an *actual infinitesimal in* $(B, +, \cdot, \leq)$, and write $a \simeq 0$, if

$$\forall n \in \mathbb{N} : \ -1 < na < 1.$$

[1]That is, the relation $\leq$ is symmetric and transitive, but not necessarily antisymmetric, as in Smooth Infinitesimal Analysis, see e.g., J. Bell [1].

The ring B is said to be *Bernoullian* if it contains at least one nonzero actual infinitesimal.

The term *Bernoullian* refers to Johann Bernoulli, who, having learned an infinitesimal methodology from Leibniz, never wavered from it. If classical logic is adopted at the meta-mathematical level, this term is equivalent to *non-Archimedean.*

For example, the ring of dual numbers $B = \mathbb{R}[\epsilon]/(\epsilon^2 = 0)$ equipped with the ordering

$$a + \epsilon b \leq \alpha + \epsilon\beta \text{ iff } a \leq \alpha$$

is a Bernoullian ring with ϵ serving as a nonzero nilpotent infinitesimal since $\epsilon^2 = 0$. Note that we have both $\epsilon \leq 0$ and $\epsilon \geq 0$ even if $\epsilon \neq 0$.

Definition 2. A *potential infinitesimal* is a null sequence $p = (p_n)_{n\in\mathbb{N}}$, i.e., a map $p : \mathbb{N} \longrightarrow \mathbb{R}$ satisfying $\lim_{n\to\infty} p_n = 0$.

A Bernoullian ring is typically constructed starting from a subset $\mathcal{S} \subseteq \mathbb{R}^{\mathbb{N}}$ of the collection $\mathbb{R}^{\mathbb{N}}$ of real sequences, and forming a quotient

$$B = \mathcal{S}/\!\sim$$

by a suitable equivalence relation $\sim$. There is therefore a natural link between actual and potential infinitesimals, given by the natural projection

$$\pi : p \in \mathcal{S} \;\mapsto\; a = [p]_\sim \in B.$$

In an abstract form, the relationship between potential and actual infinitesimals can assume two forms, one stronger than the other. In the first form, every actual infinitesimal always has *at least one* representative sequence which is a potential infinitesimal. In the second form, *every* representative sequence is required to be a potential infinitesimal. We formalize this distinction as follows.

Definition 3. Let $(B, +, \cdot, \leq)$ be a Bernoullian ring, let $\mathcal{S} \subseteq \mathbb{R}^{\mathbb{N}}$ be a collection of real sequences, and let $\pi : \mathcal{S} \longrightarrow B$ be a surjective map. We say that in B *every infinitesimal is accessible* if

$$\forall a \in B \quad \left[a \simeq 0 \;\Rightarrow\; \exists p \in \mathcal{S} : \; \pi(p) = a \text{ and } \lim_{n\to+\infty} p_n = 0 \right]. \tag{2.1}$$

Meanwhile, we say that in B *every infinitesimal is strongly accessible* if

$$\forall a \in B \; \forall p \in \mathcal{S} \quad \left[a \simeq 0 \;,\; \pi(p) = a \;\Rightarrow\; \lim_{n\to+\infty} p_n = 0 \right]. \tag{2.2}$$

Note that since π is surjective, condition (2.2) is stronger than condition (2.1).

3. Cauchy's infinitesimals: a modern viewpoint

Robinson [20] explored the possibility of using Nonstandard Analysis to interpret Cauchy's so-called "mistake" concerning convergent series of continuous functions. Robinson's approach can be thought of as an investigation of the formal relationship between potential and actual infinitesimals.

The work of Lakatos [15], Cleave [4], [5], Cutland et al. [6], and Laugwitz [16] sought to clarify this question. The notion of an accessible infinitesimal is essentially due to Cleave [4]. Cutland et al. [6] showed that a P-point ultrafilter is required in order to prove that every infinitesimal in the hyperreal field ${}^*\mathbb{R}$ is accessible (but not strongly accessible). Indeed, there are models of ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{F}$ containing infinitesimals $a \in {}^*\mathbb{R}$ which are not accessible, i.e., such that whenever we consider a representative sequence $p \in \mathbb{R}^{\mathbb{N}}$ so that $a = [p]_\sim$, the sequence p does *not* tend to zero.[2] In such model of the hyperreal field, the interaction between potential and actual infinitesimals leaves to be desired. As Cleave wrote,

> The inaccessible infinitesimals are rather elusive entities – they appear to have no role to play in analysis. The natural way of showing the existence of infinitesimals with certain properties is actually to construct a null variable [sequence] (Cleave 1972 [4]).

Moreover, the existence of a P-point ultrafilter cannot be proved in ZFC, that is using the usual axioms of set theory plus the axiom of choice. Assuming the continuum hypothesis or Martin's axiom and using transfinite induction, it is possible to prove the existence of a P-point. See Cutland et al. [6] and references therein for more details on this foundational wrinkle.

The main question of the present work is therefore whether it is possible to construct a sufficiently powerful Bernoullian ring (a model of an infinitesimal-enriched continuum) where every infinitesimal is strongly accessible, i.e., where the interaction between potential and actual infinitesimals is more satisfactory. Here "sufficiently powerful" means a model where at least some nontrivial form of Leibniz's law of continuity holds. An example of an *insufficiently* powerful ring is the ring of dual numbers mentioned above.

[2] For an inaccessible infinitesimal, a representing sequence will not even tend to zero along any subsequence supported on a set of indices which is a member of the ultrafilter used in the construction of ${}^*\mathbb{R}$.

In the first part of this text, we will use a simple ultrapower construction, but we take advantage of this to critically analyze this construction: which desired properties force us to consider an ultrafilter? Is this construction always intuitively clear? To obtain Leibniz's law, are we *forced* to consider an ultrafilter? Can we state Leibniz's law without having a background in formal Logic?

In the second part of this text, we will *not* use an ultrapower, but we will nonetheless arrive at a Bernoullian ring where every infinitesimal is *strongly accessible*. However, this second approach produces nilpotent infinitesimals and a form of Leibniz's law in intuitionistic logic. In spite of the completely different final results, the basic idea for the construction of a new model of the continuum is very similar, as explained in the next section.

4. The basic idea

Cantor's completion of the rationals resulting in the field of real numbers proceeds by quotienting the collection $\mathcal{C} \subset \mathbb{Q}^{\mathbb{N}}$ of all Cauchy sequences of rational numbers by Cauchy's equivalence relation. Similarly, the collection $\mathcal{C} \subset \mathbb{R}^{\mathbb{N}}$ of all Cauchy sequences of real numbers projects to the Archimedean continuum $\mathbb{R}$:

$$\mathcal{C} \xrightarrow{\lim} \mathbb{R}. \tag{4.1}$$

The corresponding equivalence relation,

$$u \sim_{\mathcal{C}} v \quad \text{iff} \quad \lim_{n\to\infty} |u_n - v_n| = 0,$$

"collapses" all null sequences to a single point $0 \in \mathbb{R}$. Is there another way to define an equivalence relation $\sim$ on $\mathcal{C}$ that would allow some null sequences to retain their distinct identity? In other words, can one refine Cantor's equivalence relation among Cauchy sequences, in such a way as not to "collapse" all null sequences to zero?

The idea, roughly, is to seek to retain some information in the quotient about the *rate of convergence* of a typical sequence. Then, relative to the new equivalence relation $\sim$, a null sequence of reals would become an actual infinitesimal. In other words, we are searching for a new notion of "completion", with respect to which the real field $\mathbb{R}$ can be completed by the addition of infinitesimals. What one seeks is an *intermediate stage*, $B := \mathcal{C}/\sim$ (or, more generally, a quotient $B = \mathcal{S}/\sim$, where $\mathcal{S} \subseteq \mathcal{C}$), in the projection (4.1), which would represent an infinitesimal-enriched continuum as in Figure 4.1.

Here, if $[u]_\sim$ is the equivalence class of a sequence u, then the function

$$\mathrm{st} : B \to \mathbb{R},$$

$$\begin{tikzcd} & B \arrow[d, shift left, "\mathrm{st}"] \\ \mathcal{C} \arrow[ur, "p"] \arrow[r, "\lim"'] & \mathbb{R} \arrow[u, hook, shift left] \end{tikzcd}$$

FIGURE 4.1. Factoring Cantor's map $\mathcal{C} \to \mathbb{R}$

defined by

$$\mathrm{st}([u]_\sim) := \lim_{n\to+\infty} u_n \in \mathbb{R}$$

is the usual limit of a Cauchy sequence $u = (u_n)_{n\in\mathbb{N}} \in \mathcal{C}$. This function represents the *standard part* of $[u]_\sim \in B$, that is a standard real number infinitely close to the new number $[u]_\sim \in B$. The most natural way to obtain a ring structure on B is to define the equivalence relation $\sim$ so that it preserves pointwise sums and products. We expect B to be only a ring rather than a field, because it cannot contain the pointwise inverse $\left(\frac{1}{u_n}\right)_{n\in\mathbb{N}}$ of an infinitesimal $[u_n]$, since the inverse is not a Cauchy sequence.

5. A possible approach with invertible infinitesimals

To implement the ideas outlined in Section 4, a possible approach is to declare two Cauchy sequences u, $v \in \mathcal{C}$ to be equivalent if they coincide on a "dominant" set of indices in $\mathbb{N}$:

$$u \sim v \quad \iff \quad \{n \in \mathbb{N} \mid u_n = v_n\} \text{ is dominant.} \tag{5.1}$$

For simplicity, we will use the symbol $[u]$ for the equivalence class $[u]_\sim$ generated by $u \in \mathcal{C}$.

What is "dominant"?[3] A finite set in $\mathbb{N}$ is never dominant; every cofinite set (i.e., set with finite complement) is necessarily dominant, and we also expect the property that the superset of a dominant set is dominant, as well. Moreover, we wish the relation (5.1) to yield an equivalence relation. In particular, the validity of the transitive

[3] For the reader already familiar with the usual ultrapower construction of the hyperreal field, it is clear how to formalize the idea of a dominant subset of $\mathbb{N}$, but to critically analyze this well known construction, in the following we assume that the reader is not even familiar with the notion of ultrafilter. Of course, we are not claiming that this is an alternative approach to ${}^*\mathbb{R}$ useful for teaching, because our construction is too tied to our analysis of the relationships between potential and actual infinitesimals.

property for generic Cauchy sequences implies that the intersection of two dominant sets is dominant. In fact, let us assume that

$$(\forall u, v, w \in \mathcal{C})\ (u \sim v\ \wedge\ v \sim w\ \Rightarrow\ u \sim w)\,. \tag{5.2}$$

Then, if sets A and B of indices are dominant, it suffices to take[4]

$$u_n := \begin{cases} 1 & \text{if } n \in A \\ 1 - \frac{1}{n+1} & \text{if } n \in \mathbb{N} \setminus A \end{cases} \qquad w_n := \begin{cases} 1 & \text{if } n \in B \\ 1 + \frac{1}{n+1} & \text{if } n \in \mathbb{N} \setminus B \end{cases}$$

to have that $u \sim 1$ and $1 \sim w$ so that $u \sim w$ from (5.2). This means that $\{n \in \mathbb{N} \mid u_n = w_n\} = A \cap B$ is dominant. Conversely, if our family of dominant sets is closed with respect to finite intersections, then $\sim$ is an equivalence relation. For example, the family of all cofinite sets

$$\mathcal{F} := \{S \subseteq \mathbb{N} \mid \mathbb{N} \setminus S \text{ is finite}\}\,,$$

the so-called Frechet filter, satisfies all the conditions we have imposed, up till now, on dominant sets. These conditions define the notion of a *filter* on the set $\mathbb{N}$ (extending the Frechet filter).

It is easy to prove that the equivalence relation $\sim$ preserves pointwise operations

$$[u] + [v] := \left[(u_n + v_n)_{n\in\mathbb{N}}\right] \quad \text{and} \quad [u] \cdot [v] := \left[(u_n \cdot v_n)_{n\in\mathbb{N}}\right] \tag{5.3}$$

so that ${}^{\mathrm{f}}\mathbb{R} := \mathcal{C}/\sim$ becomes a ring. The upper left index 'f' in the symbol ${}^{\mathrm{f}}\mathbb{R}$ alludes to the fact that in this ring there are no infinite numbers, but only finite ones. Whether or not ${}^{\mathrm{f}}\mathbb{R}$ is an integral domain depends on the choice of the filter of dominant sets. Note that the relation as in (5.1) is a *refinement* of the usual Cauchy relation $\sim_{\mathcal{C}}$. Indeed, if u, $v \in \mathcal{C}$ coincide on a dominant set A, then we have $u_{\sigma_n} - v_{\sigma_n} = 0$ for some subsequence $\sigma : \mathbb{N} \to \mathbb{N}$ (enumerating the members of the set A). It follows that $u \sim_{\mathcal{C}} v$ since u and v converge. Of course, the relation $\sim$ is a strict refinement because if we take $u_n = \frac{1}{n}$ and $v_n = 0$, then $u \sim_{\mathcal{C}} v$ but the collection $\{n \in \mathbb{N} \mid u_n = v_n\} = \emptyset$ is the empty set, which is never dominant.

Whether or not the idea expressed by the notion of a dominant set as in (5.1) can be considered "natural" is a matter of opinion. An alternative approach would be to define a new equivalence relation in terms of the rate of convergence of the difference $u - v$. A thread going in this direction will be presented in section 7, but here we will continue with the approach based on (5.1). If one accepts this idea, then it is also natural to define an *order*, by setting

$$[u] \geq [v] \quad\iff\quad \{n \in \mathbb{N} \mid u_n \geq v_n\} \text{ is dominant.} \tag{5.4}$$

[4]Let us note that in $\mathbb{N}$ we have $0 \in \mathbb{N}$.

This yields an ordered ring, as one can easily check.

Is this order total? The assumption that it is total, i.e.

$$\forall u \in \mathcal{C} : \; [u] \geq 0 \quad \text{or} \quad [u] \leq 0, \tag{5.5}$$

yields a further condition on dominant sets. In fact, if A is dominant, then defining

$$u_n := \begin{cases} \frac{1}{n+1} & \text{if } n \in A \\ -\frac{1}{n+1} & \text{if } n \in \mathbb{N} \setminus A \end{cases}$$

we have that A is dominant if the first alternative of (5.5) holds; otherwise $\mathbb{N} \setminus A$ is dominant. A filter verifying this additional condition is called a *free ultrafilter* and from now on we will consider such a free ultrafilter in our construction. Using this additional condition, we are now also able to prove that ${}^{\mathrm{f}}\mathbb{R}$ is an integral domain.

Theorem 4. *${}^{\mathrm{f}}\mathbb{R}$ is an integral domain.*

Proof. Given nonzero classes $[u] \neq 0$ and $[v] \neq 0$, both of the sets $\{n \in \mathbb{N} \,|\, u_n \neq 0\}$ and $\{n \in \mathbb{N} \,|\, v_n \neq 0\}$ are dominant. Therefore so is their intersection. $\square$

Given an integral domain, we can consider the corresponding field of fractions ${}^{\mathrm{f}}\mathbb{R}_{\text{frac}}$. Since ${}^{\mathrm{f}}\mathbb{R}$ is also an ordered ring, the same structure carries over to the quotient field of fractions in the usual way.

Remark 5. In a classical approach to nonstandard analysis, the equality on a dominant set (Definition (5.1)) is applied to *arbitrary* sequences, rather than merely Cauchy sequences. Nonetheless, our field of fractions ${}^{\mathrm{f}}\mathbb{R}_{\text{frac}}$ is isomorphic to the full hyperreal field ${}^*\mathbb{R}$ of nonstandard analysis through

$$\frac{[u]}{[v]} \in {}^*\mathbb{R}_{\text{frac}} \mapsto \left[\left(\frac{u_n}{v_n} \right)_{n \in \mathbb{N}} \right]_{\mathcal{U}} \in {}^*\mathbb{R},$$

where $[(q_n)_n]_{\mathcal{U}}$ is the equivalence class modulo the ultrafilter $\mathcal{U}$. To prove this, note that every sequence $q \in \mathbb{R}^{\mathbb{N}}$ can be written as $q = \frac{u}{v}$ for two null sequences u, v, e. g., taking $u_n := q_n \cdot \frac{1}{e^{|q_n|} \cdot (n+1)}$ and $v_n := \frac{1}{e^{|q_n|} \cdot (n+1)}$. This clarifies the relationship between our ${}^{\mathrm{f}}\mathbb{R}$ and the usual hyperreal field ${}^*\mathbb{R}$. It also underscores the fact that the goal of our approach is to exhibit a Bernoullian ring where every infinitesimal is strongly accessible, but this splitting of ${}^*\mathbb{R}$ into two steps

$$\mathbb{R} \hookrightarrow {}^{\mathrm{f}}\mathbb{R} \hookrightarrow {}^{\mathrm{f}}\mathbb{R}_{\text{frac}} = {}^*\mathbb{R}$$

is confusing from the didactic point of view.

In Tarski [21] we can find the proof, using Zorn's lemma, that the Frechet filter can be extended to a free ultrafilter. It is possible to prove that some form of the axiom of choice is necessary to prove the existence of a free ultrafilter. Yet, one consequence of exploiting this axiom is that we don't possess detailed information about how free ultrafilters are made. Moreover, this also implies that it is not so easy to prove the existence of a free ultrafilter satisfying some given and potentially useful conditions.

Admittedly, it is not easy to evaluate the idea (5.1). Thus, whatever the example of the ultrafilter we will be able to present, it doesn't seems sufficiently meaningful why the infinitesimal $\left[\left(\frac{(-1)^n}{n+1}\right)_{n\in\mathbb{N}}\right]$ would considered positive and not negative, or vice versa.

Moreover, examining the conditions defining the notion of an ultrafilter, one can guess that the notion of a dominant set is not that clear intuitively. In point of fact, the technically desirable conditions about the closure with respect to intersection and complement can lead to counterintuitive consequences. We would have that even numbers P_2 or odd numbers will be dominant (but not both). Let us suppose, e.g., the first case and continue: even numbers in P_2, i.e. the set P_4 of multiples of 4, or its complement $\mathbb{N} \setminus P_4$ will be dominant. In the latter case, also $P_2 \cap (\mathbb{N} \setminus P_4)$, i.e. numbers of the form $2(2n+1)$, will be dominant. In any case we would be able to find always a dominant set which has "1/2 of the elements of the previous dominant set". Continuing in this way, we can obtain a dominant set, which is intuitively very sparse with respect to its complement. To understand this idea a little better, let us consider that everything we said up to now can be generalized if instead of sequences $u : \mathbb{N} \to \mathbb{R}$ we take functions $u : [0,1] \to \mathbb{R}$. In other words, instead of taking our indices as integer numbers, we take real numbers in $[0,1]$. Then, we can repeat the previous reasoning considering, at each step k, subintervals of length 2^{-k}. Therefore, for every $\varepsilon > 0$, we are always able to find in an ultrafilter on $[0,1]$ a dominant set A whose uniform probability $P(A) < \epsilon$, whereas $P([0,1] \setminus A) > 1-\epsilon$, even though this complement is not dominant. See [8] for a formalization of this idea using the notion of density of subsets of $\mathbb{N}$, i.e. of finitely additive uniform probability to pick a number from a subset $A \subseteq \mathbb{N}$.

Summarizing, we have the feeling that the idea of requiring sequences to coincide on dominant sets, even if it seems not so clear from an intuitive point of view, appears to be formally extremely powerful.

Alternatively, in section 7 we will present another idea, which is intuitively clear but which doesn't seem equally powerful. Which thread one wishes to follow would depend on applications envisioned.

Of course, ${}^{\mathrm{f}}\mathbb{R}$ is a Bernoullian ring since e.g., $a = [1/(n+1)] \in {}^{\mathrm{f}}\mathbb{R}$ is an infinitesimal. The following results shows that in ${}^{\mathrm{f}}\mathbb{R}$ every infinitesimal is strongly accessible, of course with respect to the natural projection $\pi = [-] : \mathcal{C} \longrightarrow {}^{\mathrm{f}}\mathbb{R}$.

Theorem 6. *Let $[u] \in {}^{\mathrm{f}}\mathbb{R}$, then we have that*

$$[u] \text{ is an actual infinitesimal}$$

if and only if

$$\lim_n u_n = 0.$$

Therefore in ${}^{\mathrm{f}}\mathbb{R}$ every infinitesimal is strongly accessible.

Proof. Let us assume that $[u]$ is infinitesimal, then for each $n \in \mathbb{N}\setminus\{0\}$, the set

$$A_n := \left\{ k \in \mathbb{N} \,:\, -\frac{1}{n} < u_k < \frac{1}{n} \right\}$$

is dominant. Therefore, it is infinite and we can always find an increasing sequence $k : \mathbb{N} \to \mathbb{N}$ such that $k_n \in A_n$ and $k_{n+1} > k_n$. For such a sequence we have

$$\forall n \in \mathbb{N}_{\neq 0} : \; -\frac{1}{n} < u_{k_n} < \frac{1}{n}.$$

Thus, since $u \in \mathcal{C}$ is a Cauchy sequence, we have

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} u_{k_n} = 0.$$

To prove the converse implication, we can consider that

$$\forall n \in \mathbb{N}_{\neq 0} \, \exists N : \; \forall k \in \mathbb{N}_{\geq N} : \frac{1}{n} < u_k < \frac{1}{n}.$$

Since every cofinite set $\mathbb{N}_{\geq N}$ is dominant, this prove that $[u]$ is infinitesimal. □

The next aim of the present work is to show that the Bernoullian ring ${}^{\mathrm{f}}\mathbb{R}$ is "sufficiently powerful", i.e. that for this ring a version of Leibniz's law holds. As we mentioned above is impossible to have a Bernoullian ring constructed using an ultrapower $B = \mathcal{S}/\sim_{\mathcal{U}}$, $\mathcal{S} \subseteq \mathbb{R}^{\mathbb{N}}$, $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$, where every infinitesimal is strongly accessible and where a full Leibniz's law without limitation holds. Indeed, in that case we must have $\mathcal{S} = \mathbb{R}^{\mathbb{N}}$ and hence $B = {}^*\mathbb{R}$ is the usual hyperreal field, where infinitesimals are only weakly accessible. Therefore, in the ring ${}^{\mathrm{f}}\mathbb{R}$ we must have a limited version of Leibniz's law. This is quite natural since to preserve Cauchy sequences only continuous functions can be extended from $\mathbb{R}$ to ${}^{\mathrm{f}}\mathbb{R}$. The aim of the next section is to clarify this point, where we must introduce also the notion of continuous relation.

6. Leibniz's law of continuity in ${}^{\mathrm{f}}\mathbb{R}$

To convey the full power of the idea (5.1), we have to go back to Leibniz. In his introduction of infinitesimal and infinite quantities, he developed a heuristic principle called the "law of continuity", which had roots in the work of earlier scholars such as Nicholas of Cusa and Johannes Kepler. It is the principle that:

> What succeeds for the finite numbers succeeds also for the infinite numbers

(see Knobloch [14, p. 67], Robinson [20, p. 266], and Laugwitz [17]).

Kepler had already used it to calculate the area of the circle by representing the latter as an infinite-sided polygon with infinitesimal sides, and summing the areas of infinitely many triangles with infinitesimal bases. Leibniz used the law to extend concepts such as arithmetic operations, from ordinary numbers to infinitesimals, laying the groundwork for infinitesimal calculus.

Of course, a modern mathematical version of this heuristic law depends on our formalization of the word "what" in the law of continuity as stated above. Recall that in our notations $\mathcal{C}$ is the space of Cauchy sequences of real numbers.

Definition 7. Let $f:\mathbb{R}^d\to\mathbb{R}$ be a continuous function. Then $f\circ(u^1,\dots,u^d)\in\mathcal{C}$ for every d-tuple of Cauchy sequences $u^1,\dots,u^d\in\mathcal{C}$, and we can define the extension ${}^*f$ by setting

$$^*f\left([u^1],\dots,[u^d]\right):=\left[\left(f(u^1_n,\dots,u^d_n)\right)_{n\in\mathbb{N}}\right]_\sim \quad \forall[u^1],\dots,[u^d]\in{}^{\mathrm{f}}\mathbb{R}.$$

This gives a true extension of f, i.e. ${}^*f(r_1,\dots,r_d)=f(r_1,\dots,r_d)$ for every $r_1,\dots,r_d\in\mathbb{R}$ (identified with the corresponding constant sequences).

Theorem 8. *Let f, $g:\mathbb{R}^d\to\mathbb{R}$ be continuous functions, then it results*

$$\forall x_1,\dots,x_d\in\mathbb{R}:\ f(x_1,\dots,x_d)=g(x_1,\dots,x_d) \tag{6.1}$$

if and only if

$$\forall\alpha_1,\dots,\alpha_d\in{}^{\mathrm{f}}\mathbb{R}:\ {}^*f(\alpha_1,\dots,\alpha_d)={}^*g(\alpha_1,\dots,\alpha_d) \tag{6.2}$$

Analogously, we can formulate the transfer of inequalities of the form $f(x_1,\dots,x_d)<g(x_1,\dots,x_d)$.

Proof. The equality (6.1) implies

$$\left\{n \in \mathbb{N} \mid f(a_n^1, \dots, a_n^d) = g(a_n^1, \dots, a_n^d)\right\} = \mathbb{N},$$

where $[a^k] = \alpha_k$. The whole set $\mathbb{N}$ is dominant, and therefore (6.2) follows. The converse implication follows from the fact that ${}^*f$ and ${}^*g$ extend f and g and from the embedding $\mathbb{R} \subset {}^{\mathrm{f}}\mathbb{R}$. ☐

Can Leibniz's law of continuity be proved for more general properties, e.g. for order relations or disjunctions of equality and inequality or even more general relations? To solve this problem, we start, once again, from an historical consideration. Cauchy used infinitesimals to define continuity as follows: a function f is continuous between two bounds if for all x between those bounds, the difference $f(x+h) - f(x)$ will be infinitesimal whenever h is infinitesimal. Such a definition tends to bewilder a modern reader, used to thinking of f as being defined for real values of the variable x, but now we can think of $f(x+h)$ as corresponding to ${}^*f(x+h)$. The function f is not necessarily defined on all of $\mathbb{R}$, so that an extension of the real domain D of the function is implicit in Cauchy's construction. Therefore, we will start by defining such extension of $D \subseteq \mathbb{R}$.

Definition 9. Let $u \in \mathcal{C}$ be a Cauchy sequence and $D \subseteq \mathbb{R}$, then

(i) $u_n \in_n D$ iff the set $\{n \in \mathbb{N} \mid u_n \in D\}$ is dominant.

(ii) ${}^{\mathrm{f}}D := \left\{[u] \in {}^{\mathrm{f}}\mathbb{R} \mid u_n \in_n D\right\}$

Let us note that the variable n is mute in the notation $u_n \in_n D$.

Using this notation, our questions concerning Leibniz's law of continuity can be formulated as preservation properties of the operator ${}^{\mathrm{f}}(-)$. In fact, as in Theorem 8, where equalities between continuous functions are preserved, we can ask whether ${}^{\mathrm{f}}(-)$ preserves intersections (i.e. "and"), unions (i.e. "or"), set-theoretic difference (i.e. "not"), inclusions (i.e. "if... then..."), etc. Since we are looking for a meaningful but necessarily weaker version of Leibniz's law in ${}^{\mathrm{f}}\mathbb{R}$, it is also natural to ask whether the operator ${}^{\mathrm{f}}(-)$ preserves all the logical operations or not. To this end, it is interesting to note that a minimal set of extension properties necessarily implies ultrafilter conditions. As we will see, this is strictly related to the second part of the present work, where e.g. only the intuitionistic version of negation is preserved.
We will use a circle superscript ${}^\circ(-)$ to indicate a general extension.

Theorem 10. *Assume that ${}^\circ(-) : \mathcal{P}(\mathbb{R}) \to \mathcal{P}({}^\circ\mathbb{R})$ preserves unions, intersections and complements, i,.e. for every A, $B \subseteq \mathbb{R}$, we have*

$${}^\circ(A \cup B) = {}^\circ A \cup {}^\circ B$$

$$^{\circ}(A \cap B) = {}^{\circ}A \cap {}^{\circ}B$$

$$^{\circ}(A \setminus B) = {}^{\circ}A \setminus {}^{\circ}B.$$

Finally, let $e \in {}^{\circ}\mathbb{R}$. *Then*

$$\mathcal{R}_e := \{X \subseteq \mathbb{R} \,|\, e \in {}^{\circ}X\} \text{ is an ultrafilter on } \mathbb{R},$$

and if $e \in {}^{\circ}\mathbb{N}$, *then*

$$\mathcal{N}_e := \{X \cap \mathbb{N} \,|\, X \in \mathcal{R}_e\} \text{ is an ultrafilter on } \mathbb{N}.$$

Proof. We need first to prove that ${}^{\circ}(-)$ preserves also the empty set and inclusions. Indeed, ${}^{\circ}\emptyset = {}^{\circ}(\emptyset \setminus \emptyset) = {}^{\circ}\emptyset \setminus {}^{\circ}\emptyset = \emptyset$. Assume $A \subseteq B$, so that $A = A \cap B$ and ${}^{\circ}A = {}^{\circ}A \cap {}^{\circ}B$ and thus ${}^{\circ}A \subseteq {}^{\circ}B$.

If $X, Y \in \mathcal{R}_e$, then $e \in {}^{\circ}X \cap {}^{\circ}Y = {}^{\circ}(X \cap Y)$, and hence $X \cap Y \in \mathcal{R}_e$. If $X \in \mathcal{R}_e$ and $\mathbb{R} \supseteq Y \supseteq X$, then $e \in {}^{\circ}X \subseteq {}^{\circ}Y$ and hence $Y \in \mathcal{R}_e$. If $X \subseteq \mathbb{R}$, then $\mathbb{R} = X \cup (\mathbb{R} \setminus X)$; but $e \in {}^{\circ}\mathbb{R} = {}^{\circ}X \cup ({}^{\circ}\mathbb{R} \setminus {}^{\circ}X)$ and therefore $X \in \mathcal{R}_e$ or ${}^{\circ}\mathbb{R} \setminus {}^{\circ}X \in \mathcal{R}_e$, and this finally proves that $\mathcal{R}_e$ is an ultrafilter on $\mathbb{R}$.

The proof that $\mathcal{N}_e$ is closed with respect to intersection is direct. Consider $\mathbb{N} \supseteq S \supseteq X \cap \mathbb{N}$ with $X \in \mathcal{R}_e$; then $Y := (S \setminus X) \cup X \supseteq X$ and hence $Y \in \mathcal{R}_e$. Therefore, $Y \cap \mathbb{N} = S$ because $X \cap \mathbb{N} \subseteq S$, and hence $S \in \mathcal{N}_e$. Finally, if $S \subseteq \mathbb{N}$, then either $S \in \mathcal{R}_e$, and thus $S = S \cap \mathbb{N} \in \mathcal{N}_e$, or $\mathbb{R} \setminus S \in \mathcal{R}_e$. In the second case, $(\mathbb{R} \setminus S) \cap \mathbb{N} = \mathbb{N} \setminus S \in \mathcal{N}_e$. Up to now, we didn't need the further hypothesis $e \in {}^{\circ}\mathbb{N}$. However, in this case, if $X \in \mathcal{R}_e$, then $e \in {}^{\circ}X \cap {}^{\circ}\mathbb{N} = {}^{\circ}(X \cap \mathbb{N}) \neq \emptyset$ and hence also $X \cap \mathbb{N} \neq \emptyset$. □

Taking, e.g., $e = 1 \in {}^{\mathrm{f}}\mathbb{N}$, yields an ultrafilter.

The meaning of this theorem is the following: if one doesn't like the idea (5.1) but wants to obtain something corresponding to Leibniz's law of continuity, one must face the problem that the corresponding extension operator ${}^{\circ}(-)$ cannot preserves "and", "or" and "not" of arbitrary subsets. In section 7, where we will introduce another idea to refine Cauchy's equivalence relation without using ultrafilters, we will see that a corresponding law of continuity holds, but only for open subsets, so that we are forced to define a set-theoretical difference with values in open sets

$$A \setminus B := \mathrm{int}(A \setminus B),$$

where $\mathrm{int}(-)$ is the interior operator. Note that the use of open sets and this "not" operator correspond to the semantics of intuitionistic logic.

For the sake of completeness, we also add the following results, which represents particular cases of the previous Theorem 10.

Corollary 11. *In the hypothesis of Theorem 10, if*

$$X \subseteq {}^{\circ}X \ , \ ({}^{\circ}X \setminus X) \cap \mathbb{R} = \emptyset \quad \forall X \subseteq \mathbb{R}, \tag{6.3}$$

then we have that $e \in \mathbb{R}$ *if and only if* $\mathcal{R}_e$ *is the* principal ultrafilter generated by e*, i.e.*

$$\mathcal{R}_e = \{X \subseteq \mathbb{R} \,|\, e \in X\}\,. \tag{6.4}$$

Proof. Let us assume that $e \in \mathbb{R}$ and prove the equality (6.4). If $e \in X \subseteq \mathbb{R}$, then $e \in {}^{\circ}X$ because $X \subseteq {}^{\circ}X$ by hypotheses, and therefore $X \in \mathcal{R}_e$. Vice versa if $e \in {}^{\circ}X$, then ${}^{\circ}X = X \cup ({}^{\circ}X \setminus X)$ and hence $e \in X$ because, by hypotheses, $({}^{\circ}X \setminus X) \cap \mathbb{R} = \emptyset$ and $e \in \mathbb{R}$.

Finally, the converse implication follows directly from the equality (6.4) and from $\mathbb{R} \in \mathcal{R}_e$. □

Therefore, if the extension operator ${}^{\circ}X$ really extends X (first condition of (6.3)) adding new non real points (second condition of (6.3)), then taking $e \in \mathbb{R}$ we get a trivial ultrafilter. However, in our construction we started from a free ultrafilter; this is the case considered in the following

Corollary 12. *In the hypothesis of Corollary 11, let us assume that* $({}^{\circ}\mathbb{R}, \leq)$ *is an ordered set extending the usual order relation on the reals. Suppose that* $e \in {}^{\circ}\mathbb{R} \setminus \mathbb{R}$ *is an* infinite *with respect to* $({}^{\circ}\mathbb{R}, \leq)$*, i.e.*

$$\forall N \in \mathbb{N} : \ e > N$$

and also that

$$\forall N \in \mathbb{N} : \ e > N \ \Rightarrow \ e \in {}^{\circ}[N, +\infty),$$

then the ultrafilter $\mathcal{N}_e$ *is free.*

For example, the field ${}^{\mathrm{f}}\mathbb{R}_{\mathrm{frac}}$ verifies the hypothesis of this corollary if we take $e = \frac{[1]}{[(\frac{1}{n})_n]}$.

Proof. By our hypothesis, every interval $[N, +\infty) = \{x \in \mathbb{R} \,|\, x \geq N\}$ is in $\mathcal{R}_e$, therefore $[N, +\infty) \cap \mathbb{N} \in \mathcal{N}_e$. If $X \subseteq \mathbb{N}$ is cofinite, then $\mathbb{N} \setminus X \subseteq [0, N)$ for some $N \in \mathbb{N}$ and hence $X \supseteq [N, +\infty) \cap \mathbb{N}$. From Theorem 10, we have that $\mathcal{N}_e$ is an ultrafilter, so that it is closed with respect to supersets, and hence $X \in \mathcal{N}_e$. □

Our operator ${}^{\mathrm{f}}(-)$ has the following preservation properties of propositional logic operators.

Theorem 13. *Let A, $B \subseteq \mathbb{R}$, then the following preservation properties hold*

(i) ${}^{\mathrm{f}}(A \cup B) = {}^{\mathrm{f}}A \cup {}^{\mathrm{f}}B$
(ii) ${}^{\mathrm{f}}(A \cap B) = {}^{\mathrm{f}}A \cap {}^{\mathrm{f}}B$
(iii) ${}^{\mathrm{f}}(A \setminus B) = {}^{\mathrm{f}}A \setminus {}^{\mathrm{f}}B$
(iv) $A \subseteq B$ *if and only if* ${}^{\mathrm{f}}A \subseteq {}^{\mathrm{f}}B$
(v) ${}^{\mathrm{f}}\emptyset = \emptyset$
(vi) ${}^{\mathrm{f}}A = {}^{\mathrm{f}}B$ *if and only if* $A = B$.

Proof. For example, we will prove the preservation of unions, the other proofs being similar. Take $[u] \in {}^{\mathrm{f}}(A \cup B)$, then $\{n \mid u_n \in A \cup B\}$ is dominant. If $\{n \mid u_n \in A\}$ is dominant, then $[u] \in {}^{\mathrm{f}}A$; if not, $\{n \mid u_n \notin A\}$ is dominant and therefore it is also the intersection

$$\{n \mid u_n \in A \cup B\} \cap \{n \mid u_n \notin A\} = \{n \mid u_n \in B\},$$

so that $[u] \in {}^{\mathrm{f}}B$. Vice versa, if e.g. $[u] \in {}^{\mathrm{f}}A$, then $\{n \mid u_n \in A\}$ is dominant, and hence also the superset $\{n \mid u_n \in A \cup B\}$ is dominant, i.e. $[u] \in {}^{\mathrm{f}}(A \cup B)$. □

Example 14. Let$A$, B , $C \subseteq \mathbb{R}$ and write e.g. $A(x)$ to mean $x \in A$. We want to see that our previous Theorem 13 implies that Leibniz's law of continuity applies to complicated formulas like

$$\forall x \in \mathbb{R} : \ A(x) \ \Rightarrow \ [B(x) \text{ and } (C(x) \ \Rightarrow \ D(x))] . \tag{6.5}$$

In other words, we will show how to apply the previous theorem to show that (6.5) holds if and only if the following formula holds

$$\forall x \in {}^{\mathrm{f}}\mathbb{R} : \ {}^{\mathrm{f}}A(x) \ \Rightarrow \ \left[{}^{\mathrm{f}}B(x) \text{ and } \left({}^{\mathrm{f}}C(x) \ \Rightarrow \ {}^{\mathrm{f}}D(x)\right)\right], \tag{6.6}$$

where e.g. ${}^{\mathrm{f}}A(x)$ means $x \in {}^{\mathrm{f}}A$. In fact, if we assume (6.5), this implies that $A \subseteq B$ and hence, by Theorem 13, ${}^{\mathrm{f}}A \subseteq {}^{\mathrm{f}}B$. Therefore, if we assume ${}^{\mathrm{f}}A(x)$, for $x \in {}^{\mathrm{f}}\mathbb{R}$, from this we immediately obtain ${}^{\mathrm{f}}B(x)$. The hypotheses (6.5) also implies that $A \cap C \subseteq D$, so that if we further assume ${}^{\mathrm{f}}C(x)$ we also obtain that ${}^{\mathrm{f}}D(x)$ holds, and this concludes the proof of (6.6). Analogously we can prove the opposite implication.

Remark 15. Of course, the previous example can be generalized to every logical formula, proceeding by induction on the length of the formula, but this require the usual (simple) background of (elementary) formal logic. There is a research thread in Nonstandard Analysis (see e.g. [12, 7, 2, 8]) that tries to limit, as far as possible, the need to have a background in formal Logic to work with Nonstandard Analysis. In the present work, we also want to show that a more standard formulation of Leibniz's law by means of preservation properties of a corresponding extension operator like ${}^{\mathrm{f}}(-)$ is possible. This approach does not need a background knowledge in formal Logic.

Now, the next problem is natural: what about the preservation of existential and universal quantifier? We have already considered the case of logical connectives like "and", "or", "not" without stressing too much on the need to have a background in formal logic. These ideas could be repeated for the usual hyperreal field ${}^*\mathbb{R}$ and, in our opinion, permit to simplify the teaching of ${}^*\mathbb{R}$ and opens this type of setting to a more general audience, like physicists and engineers. We want to keep the same thread also for quantifiers. For this goal we consider two sets X, $Y \subseteq \mathbb{R}$ and the projection $p_X : X \times Y \to X$, $p_X(x, y) = x$, and $C \subseteq X \times Y$, i.e. a relation of the form $C(x, y)$ with $x \in X$ and $y \in Y$. We have

$$\begin{aligned} p_X(C) &= \{x \in X \mid \exists z \in C : \ x = p_X(z)\} = \\ &= \{x \in X \mid \exists y \in Y : \ C(x, y)\}\,; \\ X \setminus p_X\left[(X \times Y) \setminus C\right] &= \{x \in X \mid \neg\left(\exists y \in Y : \ (x, y) \notin C\right)\} = \\ &= \{x \in X \mid \forall y \in Y : \ C(x, y)\}\,. \end{aligned}$$

Therefore, now our aim is to prove that ${}^{\mathrm{f}}(-)$ preserves $p_X(C)$, which correspond to existential quantifier (preservation of universal quantifier follows from this and from the preservation of difference). Only here we notice that, exactly as we proceeded for functions considering only the continuous ones, we need an analogous condition for *relations*: what is a *continuous relation* $C \subseteq X \times Y$? To find the corresponding definition, we start from the idea that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then we expect that the relation $\{(x, y) \in X \times Y \mid y = f(x)\}$ is continuous. We can therefore note that the peculiarity of the definition of the extension ${}^*f$ (see Definition 7) is that the continuity permits to define ${}^*f$ on the whole ${}^{\mathrm{f}}\mathbb{R}$. Otherwise, we would always had the possibility to define ${}^*f$ on the smaller domain

$$\left\{[u] \in {}^{\mathrm{f}}\mathbb{R} \mid f \circ u \in \mathcal{C}\right\}.$$

For this reason, we start to define

Definition 16. Let X, $Y \subseteq \mathbb{R}$ and $C \subseteq X \times Y$, then

$${}^{\mathrm{f}}C := \left\{([u], [v]) \in {}^{\mathrm{f}}X \times {}^{\mathrm{f}}Y \mid (u_n, v_n) \in_n C\right\},$$

and we start to compare $\mathrm{dom}({}^{\mathrm{f}}C)$ and ${}^{\mathrm{f}}[\mathrm{dom}(C)]$.

Theorem 17. *In the previous hypothesis, we always have*

$$\mathrm{dom}({}^{\mathrm{f}}C) \subseteq {}^{\mathrm{f}}[\mathrm{dom}(C)]$$

$$\mathrm{cod}({}^{\mathrm{f}}C) \subseteq {}^{\mathrm{f}}[\mathrm{cod}(C)],$$

where $\mathrm{dom}(C) = \{x \in X \mid \exists y \in Y : \ C(x, y)\}$ *is the domain of* C*, and* $\mathrm{cod}(C) = \{y \in Y \mid \exists x \in X : \ C(x, y)\}$ *is the codomain of* C*.*

Proof. We prove, e.g., the relation about the domains. If $[u] \in \mathrm{dom}({}^{\mathrm{f}}C)$, then there exists v such that $([u], [v]) \in {}^{\mathrm{f}}C$, i.e. $u_n \in_n \mathrm{dom}(C)$, and this means that $[u] \in {}^{\mathrm{f}}[\mathrm{dom}(C)]$. □

Therefore, it is the opposite inclusion that represents our idea of a continuous relation.

Definition 18. In the previous hypothesis, we say that:
(i) C is continuous in the domain iff $\mathrm{dom}({}^{\mathrm{f}}C) \supseteq {}^{\mathrm{f}}[\mathrm{dom}(C)]$.
(ii) C is continuous in the codomain iff $\mathrm{cod}({}^{\mathrm{f}}C) \supseteq {}^{\mathrm{f}}[\mathrm{cod}(C)]$.

For example, in the case $C = \mathrm{graph}(f)$, the continuity in the domain says that ${}^*f$ is defined on the whole ${}^{\mathrm{f}}X$. Analogously, we can define the continuity of an n-ary relation with respect to its k-th slot.

Theorem 19. *If* X*,* $Y \subseteq \mathbb{R}$*, and* $f : X \to Y$*, then* f *is continuous if and only if graph*(f) *is continuous in the domain.*

The proof of this theorem can be directly deduced from the following consideration. The continuity of C in the domain can be written as

$$\forall u \in \mathcal{C} : \ u_n \in_n \mathrm{dom}(C) \quad \Rightarrow \quad \exists y \in {}^{\mathrm{f}}Y : {}^{\mathrm{f}}C\left([u], y\right). \tag{6.7}$$

We can write this condition in a more meaningful way if we use the notation for a generic property $\mathcal{P}(n)$:

$$\left[\forall^{\mathrm{d}} n : \ \mathcal{P}(n)\right] \quad :\Longleftrightarrow \quad \{n \in \mathbb{N} \mid \mathcal{P}(n)\} \text{ is dominant.}$$

For example, $u_n \in_n D$ can now be written as $\forall^{\mathrm{d}} n : \ u_n \in D$. Therefore, (6.7) can be written as

$$\forall u \in \mathcal{C} : \left(\forall^{\mathrm{d}} n \, \exists y \in Y : \ C(u_n, y)\right) \ \Rightarrow \ \exists y \in {}^{\mathrm{f}}Y : \ {}^{\mathrm{f}}C([u], y). \tag{6.8}$$

This can be meaningfully interpreted in the following way: if we are able to solve the equation

$$C(u_n, y_n) = \text{true}$$

finding a solution $y_n \in Y$ for a dominant set of indices n, then we are also able to solve the equation

$${}^{\mathrm{f}}C([u], y) = \text{true}$$

for a solution $y \in {}^{\mathrm{f}}Y$.
Using this formulation, it is not hard to prove that all the relations $=$, $<$ and $\leq$ are continuous both in the domain and in the codomain.

An expected example of non continuous relation is $x \cdot y = 1$ (take, e.g., $u_n := \frac{1}{n+1}$ in (6.8)). This corresponds to the non applicability of Leibniz's law of continuity to the field property

$$\forall x \in \mathbb{R} : \ x \neq 0 \ \Rightarrow \ \exists y \in \mathbb{R} : \ x \cdot y = 1,$$

which cannot be transferred to our ${}^{\mathrm{f}}\mathbb{R}$, which is only a ring and not a field.

Now, we can formulate the preservation of quantifiers:

Theorem 20. *Let X, $Y \subseteq \mathbb{R}$ and $C \subseteq X \times Y$ be a relation continuous in the domain, then*

$${}^{\mathrm{f}}[p_X(C)] = p_{{}^{\mathrm{f}}X}({}^{\mathrm{f}}C).$$

That is

$${}^{\mathrm{f}}\{x \in X \mid \exists y \in Y : \ C(x,y)\} = \left\{x \in {}^{\mathrm{f}}X \mid \exists y \in {}^{\mathrm{f}}Y : \ {}^{\mathrm{f}}C(x,y)\right\}.$$

As a consequence we also have

$${}^{\mathrm{f}}\{x \in X \mid \forall y \in Y : \ C(x,y)\} = \left\{x \in {}^{\mathrm{f}}X \mid \forall y \in {}^{\mathrm{f}}Y : \ {}^{\mathrm{f}}C(x,y)\right\}.$$

Proof. If $[u] \in {}^{\mathrm{f}}[p_X(C)]$, then $u_n \in_n p_X(C)$, i.e.

$$\forall^{\mathrm{d}} n : \ u_n \in X \ , \ \exists y \in Y : \ C(u_n, y),$$

that is the set of $n \in \mathbb{N}$ verifying this relation is dominant. This implies that $u_n \in_n X$ and hence $[u] \in {}^{\mathrm{f}}X$ and $u_n \in_n \mathrm{dom}(C)$, i.e. $[u] \in {}^{\mathrm{f}}[\mathrm{dom}(C)]$. Our relation C is continuous, so that $[u] \in \mathrm{dom}({}^{\mathrm{f}}C)$, i.e.

$$\exists \beta \in {}^{\mathrm{f}}Y : \ {}^{\mathrm{f}}C([u], \beta),$$

which can also be written as $[u] \in p_{{}^{\mathrm{f}}X}({}^{\mathrm{f}}C)$. To prove the opposite inclusion it suffices to reverse this deduction and use Theorem 17 instead of the Definition 18 of continuous relation. □

Example 21. Let us apply our transfer theorems to a sentence of the form

$$\forall a \in A \, \exists b \in B : \ C(a,b) \tag{6.9}$$

showing that it is equivalent to

$$\forall a \in {}^{\mathrm{f}}A \, \exists b \in {}^{\mathrm{f}}B : \ {}^{\mathrm{f}}C(a,b), \tag{6.10}$$

where $C \subseteq A \times B$ is a binary continuous relation. Assume (6.9) and $a \in {}^{\mathrm{f}}A$. From (6.9) we have that

$$A \subseteq \{a \in \mathbb{R} \mid \exists b \in B : \ C(a,b)\},$$

Therefore, by Theorems 13, 20, we have

$${}^{\mathrm{f}}A \subseteq \left\{a \in {}^{\mathrm{f}}\mathbb{R} \mid \exists b \in {}^{\mathrm{f}}B : \ {}^{\mathrm{f}}C(a,b)\right\}$$

and we obtain the existence of a $b \in {}^{\mathrm{f}}B$ such that ${}^{\mathrm{f}}C(a,b)$. To prove the opposite implication, it suffices to revert this deduction.

Considering the example (6.9) we recognize that proving Leibniz's law in ${}^{\mathrm{f}}\mathbb{R}$ for continuous relations, we also solved the problem: for what type of relations C can we apply Leibniz's law preserving the good dialectic between potential and actual infinitesimals we have in the ring ${}^{\mathrm{f}}\mathbb{R}$? In other words: for what formulae do we have that applying Leibniz's law we still obtain hyperreal numbers generated by Cauchy's sequences?

7. A possible approach with nilpotent infinitesimals

Another possible way of refining Cantor equivalence relation on real Cauchy sequences avoiding ultrafilters is to compare two sequences u, $v \in \mathcal{C}$ with a basic infinitesimal, e.g. $\left(\frac{1}{n}\right)_n$. We therefore set by definition

$$u \sim v \quad \Longleftrightarrow \quad \lim_{n\to+\infty} n \cdot (u_n - v_n) = 0. \tag{7.1}$$

In other words, using Landau's little-oh notation, the two Cauchy sequences are to be equivalent if

$$u_n = v_n + o\left(\tfrac{1}{n}\right) \quad \text{for } n \to +\infty.$$

Like in the previous part of the article, we will denote the equivalence class of a sequence u simply by $[u]$. The relation defined in (7.1) is stronger than the usual Cauchy relation:

$$u \sim v \quad \Rightarrow \quad \exists \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n =: \mathrm{st}([u]) \in \mathbb{R}.$$

It is also strictly stronger, because, e.g., the equivalence class $\left[\left(\frac{1}{n^p}\right)_n\right]$, with $0 < p \le 1$, is a nonzero infinitesimal. For example, the infinitesimal $\left[\left(\frac{1}{n}\right)_n\right]$ is not zero, but we can think of it as being so small that its square is zero: $\left[\left(\frac{1}{n^2}\right)_n\right] = [0]$. With respect to pointwise operations, we thus obtain a ring rather than a field. A ring with nilpotent elements may seem unwieldy; however, this was surely not the case for geometers like S. Lie, E. Cartan, A. Grothendieck, or for physicists like P.A.M. Dirac or A. Einstein (see, e.g., references in [8]). The latter used to write formulas, if $v/c \ll 1$, like

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{v^2}{2c^2}$$

containing an equality sign rather than an approximate equality sign. More generally, in [3] A. Einstein wrote

$$f(x, t+\tau) = f(x, t) + \tau \cdot \frac{\partial f}{\partial t}(x, t) \tag{7.2}$$

justifying it with the words "*since τ is very small*". Let us note that if we apply (7.2) to the function $f(x,t) = t^2$ at $t = 0$, we obtain $\tau^2 = 0 + \tau \cdot 0 = 0$ and therefore we necessarily obtain that our ring of scalars contains nilsquare elements. Of course, it is not easy to state that physicists like A. Einstein or P.A.M. Dirac were conscious to work with such kind of scalars; indeed, their work, even if sometimes is lacking from the formal/syntactical point of view, it is always strongly supported by a dialectic with the physical meaning of the discovered relationships.

A difficult point in working with a ring having nilpotent elements is the concrete management of powers of nilpotent elements, like $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}$. Let us note that this kind of products appears naturally in several variables Taylor formulae. Is this product zero or not? Are we able to decide effectively whether it is zero starting from the properties of the infinitesimals h_j and the exponents i_j? To answer positively to this, and several other questions, we restrict this construction to a particular subclass of Cauchy sequences:

Definition 22. We say that u is a *little-oh polynomial*, and we write $u \in \mathbb{R}_o\left[\frac{1}{n}\right]$ iff we can write

$$u_n = r + \sum_{i=1}^{k} \alpha_i \cdot \frac{1}{n^{a_i}} + o\left(\frac{1}{n}\right) \quad \text{as} \quad n \to +\infty, \tag{7.3}$$

for suitable $k \in \mathbb{N}$, $r, \alpha_1, \ldots, \alpha_k \in \mathbb{R}$, $a_1, \ldots, a_k \in \mathbb{R}_{\geq 0}$.

Therefore, $\mathbb{R}_o\left[\frac{1}{n}\right] \subset \mathcal{C}$ and our previous example $\left[\left(\frac{1}{n^p}\right)_n\right]$ is generated by a little-oh polynomial. Little-oh polynomials are closed with respect to pointwise ring operations, and the corresponding quotient ring

$${}^\bullet\mathbb{R} := \mathbb{R}_o\left[\frac{1}{n}\right] / \sim$$

is called ring of *Fermat reals*. The name is motivated essentially by two reasons: in the ring of Fermat reals a perfect formalization of the informal method used by A. Fermat to find derivatives is possible, see [11]; all the theory of Fermat reals and Fermat extensions has been constructed trying always to have a full dialectic between formal properties and informal geometrical interpretation: we think that this has been one of the leading methods used by A. Fermat in his work. In this section we will prove only theorems concerning actual and potential infinitesimals and Leibniz's law in ${}^\bullet\mathbb{R}$. For all the other proofs concerning the presentation of this ring, we refer to [10, 9, 11, 8].

It is not hard to prove that all the numbers k, r, α_i, a_i appearing in (7.3) are uniquely determined if we impose them the constraints

$$0 < a_1 \le a_2 \le \dots \le a_k \le 1 \tag{7.4}$$

$$\alpha_i \ne 0 \quad \forall i = 1, \dots, k. \tag{7.5}$$

We can hence introduce the following notation

Definition 23. If $x := [u] \in {}^\bullet\mathbb{R}$ and k, r, α_i, a_i are the unique real numbers appearing in (7.3) and satisfying (7.4) and (7.5), then we set ${}^\circ x := \mathrm{st}(x) := r$, ${}^\circ x_i := \alpha_i$, $\omega(x) := \frac{1}{a_1}$, $\omega_i(x) := \frac{1}{a_i}$, $N_x := k$. Moreover, we set

$$\mathrm{d}t_a := \left[\left(\frac{1}{\sqrt[a]{n}} \right)_n \right] \in {}^\bullet\mathbb{R} \quad \forall a \in \mathbb{R}_{\ge 1}$$

and, more simply, $\mathrm{d}t := \mathrm{d}t_1$. Using these notations, we can write any Fermat real as

$$x = {}^\circ x + \sum_{i=1}^{N_x} {}^\circ x_i \cdot \mathrm{d}t_{\omega_i(x)} \tag{7.6}$$

where the equality sign has to be meant in ${}^\bullet\mathbb{R}$. The numbers ${}^\circ x_i$ are called the *standard parts* of x and the numbers $\omega_i(x)$ the *orders* of x (for $i = 1$ we will simply use the names *standard part* and *order* for ${}^\circ x$ and $\omega(x)$). The unique writing (7.6) is called the *decomposition of* x.

Let us note the following properties of the infinitesimals of the form $\mathrm{d}t_a$:

$$\begin{aligned} \mathrm{d}t_a \cdot \mathrm{d}t_b &= \mathrm{d}t_{\frac{ab}{a+b}} \\ (\mathrm{d}t_a)^p &= \mathrm{d}t_{\frac{a}{p}} \quad \forall p \in \mathbb{R}_{\ge 1} \\ \mathrm{d}t_a &= 0 \quad \forall a \in \mathbb{R}_{<1}. \end{aligned}$$

A first justification to the name "order" is given by the following

Theorem 24. *If $x \in {}^\bullet\mathbb{R}$ and $k \in \mathbb{N}_{>1}$, then $x^k = 0$ in ${}^\bullet\mathbb{R}$ if and only if ${}^\circ x = 0$ and $\omega(x) < k$.*

This motivates also the definition of the following ideal of infinitesimals:

Definition 25. If $a \in \mathbb{R} \cup \{\infty\}$, then

$$D_a := \{ x \in {}^\bullet\mathbb{R} \mid {}^\circ x = 0 \ , \ \omega(x) < a + 1 \} .$$

These ideals are naturally tied with the infinitesimal Taylor formula (i.e. without any rest because of the use of nilpotent infinitesimal increments), as one can guess from the property

$$a \in \mathbb{N} \quad \Rightarrow \quad D_a = \left\{ x \in {}^\bullet\mathbb{R} \mid x^{a+1} = 0 \right\}.$$

Products of powers of nilpotent infinitesimals can be effectively decided using the following result

Theorem 26. *Let* $h_1, \dots, h_n \in D_\infty \setminus \{0\}$ *and* $i_1, \dots, i_n \in \mathbb{N}$*, then*

(i) $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = 0 \iff \sum_{k=1}^n \frac{i_k}{\omega(h_k)} > 1$

(ii) $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \neq 0 \;\Rightarrow\; \frac{1}{\omega\left(h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}\right)} = \sum_{k=1}^n \frac{i_k}{\omega(h_k)}$.

This result motivates strongly our choice to restrict our construction to little-oh polynomials only.

The reader can naturally ask what would happen in case of a different choice of the basic infinitesimal $\left(\frac{1}{n}\right)_n$ in the Definition (7.1). Really, any other choice of a different infinitesimal $(s_n)_n$ will conduct to an isomorphic ring through the isomorphism

$$^\circ x + \sum_{i=1}^{N_x} {}^\circ x_i \cdot \mathrm{d}t_{\omega_i(x)} \mapsto \left[\left(^\circ x + \sum_{i=1}^{N_x} {}^\circ x_i \cdot s_n^{\frac{1}{\omega_i(x)}} \right)_n \right]_\sim$$

This is the only ring isomorphism preserving the basic infinitesimals $\mathrm{d}t_a$ and the standard part function, i.e. such that:

$$f\left(\alpha \cdot \mathrm{d}t_a\right) = \alpha \cdot \left[\left(\sqrt[a]{s_n} \right)_n \right]_\sim$$
$$f(^\circ x) = {}^\circ f(x).$$

Essentially the same isomorphism applies also to the ring defined in [10], where instead of sequences, the construction is based on real functions of the form $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$.

7.1. **Order relation.** It is not hard to define an intuitively meaningful order relation on the ring of Fermat reals

Definition 27. Let $x, y \in {}^\bullet\mathbb{R}$ be Fermat reals, then we say that $x \leq y$ iff we can find representatives $[u] = x$ and $[v] = y$ such that

$$\exists N \in \mathbb{N}\, \forall n \geq N : \; u_n \leq v_n.$$

For all the proofs of this section, see e.g. [9, 8].

It is not hard to show that this relation is well defined on ${}^\bullet\mathbb{R}$ and that the induced order relation is total. This is another strong motivation for the choice of little-oh polynomials in the construction of the ring of Fermat reals. The analogous of Theorem 6 is the following

Theorem 28. *Let* $h \in {}^\bullet\mathbb{R}$*, then the following are equivalent* *(i)* $h \in D_\infty$, *i.e.* $^\circ h = 0$

(ii) $\forall n \in \mathbb{N}_{>0} : \; -\frac{1}{n} < h < \frac{1}{n}$, *i.e.* h *is an actual infinitesimal.*

Since if $h = [h_n]$ *then* $^\circ h = \lim_{n \to +\infty} h_n$, *in* $^\bullet\mathbb{R}$ *every infinitesimal is strongly accessible.*

Proof. From Definition 25 it follows directly that $h \in D_\infty$ iff $^\circ h = 0$. Since for every representative little-oh polynomial $^\circ h = \lim_{n \to +\infty} h_n$, we have $^\circ h = 0$ iff for all $n \in \mathbb{N}_{>0}$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$ the inequalities $-\frac{1}{n} \leq h_k \leq \frac{1}{n}$ hold. From the definition of order in $^\bullet\mathbb{R}$ this yields the conclusion. □

Therefore, it is not necessary to use an ultrapower construction to have a ring where every infinitesimal is strongly accessible.

The ring of Fermat reals $^\bullet\mathbb{R}$ is geometrically representable, [9]; it is also strongly constructive, so that a corresponding computer implementation is possible, see [12].

7.2. **Infinitesimal Taylor formula.** What kind of functions $f : \mathbb{R} \to \mathbb{R}$ can be extended on $^\bullet\mathbb{R}$? The idea for the definition of extension is natural $^\bullet f([u]) := [f \circ u]$ so that we have to chose f so that:

(1) If u is a little-oh polynomial, then also $f \circ u$ is a little-oh polynomial.
(2) If $[u] = [v]$, then also $[f \circ u] = [f \circ v]$.

The second condition is surely verified if we take f locally Lipschitz, but the first one holds if f is smooth.

Definition 29. Let $f \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ be a smooth function, then

$$^\bullet f([u^1], \ldots, [u^d]) := [f(u^1_n, \ldots, u^d_n)_n] \quad \forall [u^1], \ldots, [u^d] \in {}^\bullet\mathbb{R}.$$

Therefore, the ring of Fermat real seems potentially useful e.g. for smooth differential geometry (see e.g. chapter 13 of [8]) or in some part of physics (see e.g. [9]), where one can suppose to deal only with smooth functions.

In several applications, the following infinitesimal Taylor formulae permit to formalize perfectly the informal results frequently appearing in physics.

Theorem 30. *Let* $x \in \mathbb{R}$ *and* $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ *be a smooth function, then*

$$\exists!\, m \in \mathbb{R} \; \forall h \in D_1 : \; f(x+h) = f(x) + h \cdot m. \tag{7.7}$$

In this case we have $m = f'(x)$, *where* $f'(x)$ *is the usual derivative of* f *at* x.

Theorem 31. *Let $x \in \mathbb{R}^d$, $n \in \mathbb{N}_{>0}$ and $f \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ be a smooth function, then*

$$\forall h \in D_n^d : \; f(x+h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \le n}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|} f}{\partial x^j}(x).$$

Note that $m = f'(x) \in \mathbb{R}$, i.e. the slope is a standard real number, and that we can use the previous formula with standard real numbers x only, and not with a generic $x \in {}^\bullet\mathbb{R}$, but it is possible to remove these limitations (see [11, 9, 8]). For a generalization of these infinitesimal Taylor formalae to fractional derivatives, see [12].

8. Leibniz's law of continuity in ${}^\bullet\mathbb{R}$

Is the ring of Fermat reals sufficiently powerful? I.e. is a suitable form of the Leibniz's law of continuity provable in the ring of Fermat reals? The first version is the transfer for equality and inequality, that can be proved proceeding like in Theorem 8.

Theorem 32. *Let f, $g : \mathbb{R}^d \to \mathbb{R}$ be smooth functions, then we have*

$$\forall x_1, \ldots, x_d \in \mathbb{R} : \; f(x_1, \ldots, x_d) = g(x_1, \ldots, x_d)$$

if and only if

$$\forall x_1, \ldots, x_d \in {}^\bullet\mathbb{R} : \; {}^\bullet f(x_1, \ldots, x_d) = {}^\bullet g(x_1, \ldots, x_d).$$

Analogously, we can formulate the transfer of inequalities of the form $f(x_1, \ldots, x_d) < g(x_1, \ldots, x_d)$.

Now, we can proceed as for ${}^{\mathrm{f}}\mathbb{R}$. We firstly define the extension ${}^\bullet U$ of a generic subset $U \subseteq \mathbb{R}$.

Definition 33. Define the set of little-oh polynomials $U_o\left[\frac{1}{n}\right]$ as in Definition 22 but taking sequences $u : \mathbb{N} \to U$ with values in U and such that ${}^\circ[u] := \lim_{n \to +\infty} u_n \in U$. For u, $v \in U_o\left[\frac{1}{n}\right]$ define $u \sim v$ for $u_n = v_n + o\left(\frac{1}{n}\right)$ as $n \to +\infty$ and ${}^\bullet U := U_o\left[\frac{1}{n}\right] / \sim$.

If $i : U \hookrightarrow \mathbb{R}$ is the inclusion map, it is easy to prove that its Fermat extension ${}^\bullet i : {}^\bullet U \to {}^\bullet\mathbb{R}$ is injective. We will always identify ${}^\bullet U$ with ${}^\bullet i({}^\bullet U)$, so we simply write ${}^\bullet U \subseteq {}^\bullet\mathbb{R}$. According to this identification, if U is open in $\mathbb{R}$, we can also prove that

$${}^\bullet U = \{x \in {}^\bullet\mathbb{R} \mid {}^\circ x \in U\}. \tag{8.1}$$

Because of our Theorem 10 we must expect that our extension operator ${}^\bullet(-)$ doesn't preserve all the operators of propositional logic like "and", "or" and "not". To guess what kind of preservation properties hold for this operator we say that the theory of Fermat reals is strongly inspired by synthetic differential geometry (SDG; see, e.g., [11, 15, 1]). SDG is the most beautiful and powerful theory of nilpotent infinitesimals with important applications to differential geometry of both finite and infinite dimensional spaces. Its model require a certain knowledge of Topos theory, because a model in classical logic is not possible. Indeed, the internal logic of its topos models is necessarily intuitionistic. Fermat reals have several analogies with SDG even if, at the end it is a completely different theory. For example, in ${}^\bullet\mathbb{R}$ the product of any two first order infinitesimals is always zero, whereas in SDG this is not the case. On the other hand, the intuitive interpretation of Fermat reals is stronger and there is full compatibility with classical logic.

This background explain why we will show that our extension operator preserves intuitionistic logical operations. Even if the theory of Fermat reals can be freely studied in classical logic[5], the "most natural logic" of smooth spaces and smooth functions remains the intuitionistic one. We simply recall here that the intuitionistic Topos models of SDG show formally that L.E.J. Brouwer's idea of the impossibility to define a non smooth functions without using the law of excluded middle or the axiom of choice is correct.

Because we need to talk of open sets both in $\mathbb{R}$ and in ${}^\bullet\mathbb{R}$ we have to introduce the following

Definition 34. We always think on ${}^\bullet\mathbb{R}$ the so-called *Fermat topology*, i.e. the topology generated by subsets of the form ${}^\bullet U \subseteq {}^\bullet\mathbb{R}$ for U open in $\mathbb{R}$.

Theorem 35. *Let A, B be open sets of $\mathbb{R}$, then the following preservation properties hold*

(i) ${}^\bullet(A \cup B) = {}^\bullet A \cup {}^\bullet B$
(ii) ${}^\bullet(A \cap B) = {}^\bullet A \cap {}^\bullet B$
(iii) ${}^\bullet\mathrm{int}(A \setminus B) = \mathrm{int}({}^\bullet A \setminus {}^\bullet B)$
(iv) $A \subseteq B$ *if and only if* ${}^\bullet A \subseteq {}^\bullet B$
(v) ${}^\bullet\emptyset = \emptyset$
(vi) ${}^\bullet A = {}^\bullet B$ *if and only if* $A = B$

Proof. We will use frequently the characterization (8.1). To prove ((i)) we have that $x \in {}^\bullet(A \cup B)$ iff $x \in {}^\bullet\mathbb{R}$ and ${}^\circ x \in A \cup B$, i.e. iff ${}^\circ x \in A$ or ${}^\circ x \in B$ and, using again (8.1), this happens iff $x \in {}^\bullet A$ or $x \in {}^\bullet B$. Analogously, we can prove ((ii)). We firstly prove ((iv)). If $A \subseteq B$ and $x \in {}^\bullet A$, then ${}^\circ x \in A$ and hence also ${}^\circ x \in B$ and $x \in {}^\bullet B$. Viceversa if ${}^\bullet A \subseteq {}^\bullet B$ and $a \in A$, then ${}^\circ a = a$ so that $a \in {}^\bullet B$, that is ${}^\circ a = a \in B$.

[5] More generally, without requiring a background in formal logic.

To prove ((iii)) we have that $x \in {}^\bullet\mathrm{int}(A \setminus B)$ iff ${}^\circ x \in \mathrm{int}(A \setminus B)$, i.e iff $({}^\circ x - \delta, {}^\circ x - \delta) \subseteq A \setminus B$ for some $\delta \in \mathbb{R}_{>0}$. From ((iv)) we have ${}^\bullet({}^\circ x - \delta, {}^\circ x - \delta) \subseteq {}^\bullet A$ and $x \in {}^\bullet({}^\circ x - \delta, {}^\circ x - \delta)$. Finally, a generic $y \in {}^\bullet({}^\circ x - \delta, {}^\circ x - \delta)$ cannot belong to ${}^\bullet B$ because, otherwise, ${}^\circ y \in ({}^\circ x - \delta, {}^\circ x - \delta) \cap B$ which is impossible. Therefore, x is internal to ${}^\bullet A \setminus {}^\bullet B$ with respect to the Fermat topology. The proofs of ((v)) and ((vi)) are direct or follow directly from ((iv)). $\square$

Example 36. Using the previous theorem, we can prove the transfer of the analogue of (6.5), but where we need now to suppose that A, B, C are open subsets of $\mathbb{R}$. Therefore, we have

$$\forall x \in \mathbb{R} : \ A(x) \ \Rightarrow \ [B(x) \text{ and } (C(x) \ \Rightarrow \ D(x))]$$

if and only if

$$\forall x \in {}^\bullet\mathbb{R} : \ {}^\bullet A(x) \ \Rightarrow \ [{}^\bullet B(x) \text{ and } ({}^\bullet C(x) \ \Rightarrow \ {}^\bullet D(x))] \, .$$

Once again, we don't strictly need a background of intuitionistic logic to understand that the preservation of quantifier for the Fermat extension ${}^\bullet(-)$ must be formulated in the following way

Theorem 37. *Let A, B be open subsets of $\mathbb{R}$, and C be open in $A \times B$. Let $p : (a, b) \in A \times B \mapsto a \in A$ be the projection on the first component. Define*

$$\begin{aligned} {}^\bullet C &:= \{(\alpha, \beta) \,|\, ({}^\circ\alpha, {}^\circ\beta) \in C\} \\ \exists_p(C) &:= p(C) \\ \forall_p(C) &:= \mathrm{int}\left(A \setminus \exists_p \left(\mathrm{int}\left((A \times B) \setminus C\right)\right)\right) . \end{aligned}$$

Then

$$\begin{aligned} {}^\bullet \left[\exists_p(C)\right] &= \exists_{{}^\bullet p}({}^\bullet C) \\ {}^\bullet \left[\forall_p(C)\right] &= \forall_{{}^\bullet p}({}^\bullet C). \end{aligned}$$

That is

$$\begin{aligned} {}^\bullet \{a \in A \,|\, \exists b \in B : \ C(a, b)\} &= \{a \in {}^\bullet A \,|\, \exists b \in {}^\bullet B : \ {}^\bullet C(a, b)\} \\ {}^\bullet \{a \in A \,|\, \forall b \in B : \ C(a, b)\} &= \{a \in {}^\bullet A \,|\, \forall b \in {}^\bullet B : \ {}^\bullet C(a, b)\} . \end{aligned}$$

Proof. The preservation of the universal quantifier follows from that of the existential quantifier and from property ((iii)) of 35, so that we only have to prove ${}^\bullet [p(C)] = {}^\bullet p({}^\bullet C)$. Consider that the projection is an open map, so that $p(C)$ is open because C is open in $A \times B$. Therefore $x \in {}^\bullet [p(C)]$ iff ${}^\circ x \in p(C)$, and this holds iff we can find $(a, b) \in C$ such that ${}^\circ x = p(a, b) = a \in A$. Therefore, ${}^\bullet p(x, b) = [(p(x_n, b))_n] = [(x_n)_n] = x$ and $(x, b) \in {}^\bullet C$ because $({}^\circ x, b) = (a, b) \in C$. This proves that ${}^\bullet [p(C)] \subseteq {}^\bullet p({}^\bullet C)$. Vice versa, if $x \in {}^\bullet p({}^\bullet C)$, then we can find

$(\alpha, \beta) \in {}^\bullet C$ such that $x = {}^\bullet p(\alpha, \beta) = \alpha$. Therefore, $({}^\circ\alpha, {}^\circ\beta) \in C$ and $p({}^\circ\alpha, {}^\circ\beta) = {}^\circ\alpha = {}^\circ x$. This means that ${}^\circ x \in p(C)$, which is open and hence $x \in {}^\bullet p(C)$. $\square$

Example 38. Using the previous theorem, we can prove the transfer of the analogous of Example 21, but where we need now to suppose that A, B are open subsets of $\mathbb{R}$ and C is open in $A \times B$. Therefore, we have

$$\forall a \in A \, \exists b \in B : \ C(a, b)$$

if and only if

$$\forall a \in {}^\bullet A \, \exists b \in {}^\bullet B : \ {}^\bullet C(a, b).$$

The theory of Fermat reals can be greatly developed: any smooth manifold can be extended with similar infinitely near points and the extension functor ${}^\bullet(-)$ has wonderful preservation properties that generalize what we have just seen on the (intuitionistic) Leibniz's law of continuity in ${}^\bullet\mathbb{R}$. Potential useful applications are in the differential geometry of spaces of functions, like the space of all the smooth functions between two manifolds.

9. Conclusion

We have explored two ideas toward refining Cauchy's equivalence relation among Cauchy real sequences so as to obtain a new infinitesimal-enriched continuum where every infinitesimal is strongly accessible. The first one takes us toward a subring of the hyperreal field of non-standard analysis, and we sought to motivate all the steps one must take to arrive at a powerful theory. On the other hand, we have seen that sometimes the intuitive interpretation of these steps is lacking. The second idea is intuitively clear but surely formally less powerful. They serve different scopes because they deal with different kinds of infinitesimals: invertible and nilpotent. In both cases, the goal of having a good interaction between potential and actual infinitesimals forces one to obtain only a reduced version of the Leibniz's continuity law. Since Cauchy never worked with non-convergent bounded sequences, our work underscores the fact that interpreting Cauchy's continuum with the hyperreals ${}^*\mathbb{R}$ is questionable since in this field infinitesimals are at best weakly accessible.

References

[1] J.L. Bell, *A Primer of Infinitesimal Analysis*, Cambridge University Press, 1998.

[2] V. Benci and M. Di Nasso. A purely algebraic characterization of the hyperreal numbers. *Proceedings of the American Mathematical Society*, 133(9):2501–05, 2005.

[3] Borovik, A.; Katz, M.: Who gave you the Cauchy–Weierstrass tale? The dual history of rigorous calculus. *Foundations of Science* **17** (2012), no. 3, 245-276. See http://dx.doi.org/10.1007/s10699-011-9235-x
and http://arxiv.org/abs/1108.2885

[4] J. Cleave, Cauchy, Convergence and Continuity; The British Journal for the Philosophy of Science, 22, pp. 27-37, 1972.

[5] J. Cleave, The concept of variable in nineteenth century analysis, The British Journal for the Philosophy of Science, 30, pp. 266-77, 1979.

[6] N. Cutland, C. Kessler, E. Kopp, and D. Ross. On Cauchy's notion of infinitesimal. *British J. Philos. Sci.*, 39(3):375–378, 1988.

[7] P. Ehrlich. The rise of non-Archimedean mathematics and the roots of a misconception I: The emergence of non-Archimedean systems of magnitudes. *Archive for History of Exact Sciences*, 60(1):1–121, 2006.

[3] A. Einstein. *Investigations on the Theory of the Brownian Movement*. Dover, 1926.

[8] P. Giordano. Fermat reals: Nilpotent infinitesimals and infinite dimensional spaces. arXiv:0907.1872, July 2009.

[9] P. Giordano. Infinitesimals without logic. *Russian Journal of Mathematical Physics*, 17(2):159–191, 2010.

[10] P. Giordano. The ring of fermat reals. *Advances in Mathematics*, 225(4):2050–2075, 2010.

[11] P. Giordano. Fermat-Reyes method in the ring of Fermat reals. *Advances in Mathematics* 228, pp. 862-893, 2011.

[12] Giordano P., Kunzinger M. "Topological and algebraic structures on the ring of Fermat reals". *Israel Journal of Mathematics*, 2011, DOI: 10.1007/s11856-012-0079-z.

[7] C.W. Henson. Foundations of nonstandard analysis. A gentle introduction to nonstandard extension. In L.O. Arkeryd, N.J. Cutland, and C.W. Henson, editors, *Nonstandard analysis: theory and applications (Edinburgh, 1996)*, pages 1–49, Dordrecht, 1997. NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., vol. 493, Kluwer Acad. Publ.

[13] E. Hewitt. Rings of real-valued continuous functions. i. *Trans. Amer. Math. Soc.*, 64:45–99, 1948.

[8] J.H. Keisler. *Elementary Calculus: An Approach Using Infinitesimals*. Prindle Weber & Schmidt, 1976.

[14] E. Knobloch. Leibniz's rigorous foundation of infinitesimal geometry by means of Riemannian sums. Foundations of the formal sciences, 1 (Berlin, 1999). *Synthese* **133** (2002), no. 1-2, 59–73.

[11] A. Kock. *Synthetic Differential Geometry*, volume 51 of London Math. Soc. Lect. Note Series. Cambridge Univ. Press, 1981.

[15] I. Lakatos, The significance of Non-standard Analysis for the History and Philosophy of Mathematics, in Imre Lakatos: *Philosophical papers, Volume 2*. Cambridge University Press, 1978.

[16] D. Laugwitz, Definite values of infinite sums: aspects of the foundations of infinitesimal analysis around 1820. *Arch. Hist. Exact Sci.* **39** (1989), no. 3, 195–245.

[17] D. Laugwitz. Leibniz' principle and omega calculus. [A] Le labyrinthe du continu, Colloq., Cerisy-la-Salle/Fr. 1990, 144-154 (1992).

[18] J. Łoś. *Quelques remarques, thormes et problmes sur les classes dfinissables d'algbres*, volume Mathematical interpretation of formal systems, pages 98–113. North-Holland Publishing Co., Amsterdam, 1955.

[19] W.A.J. Luxemburg. Reduced powers of the real number system and equivalents of the Hahn-Banach extension theorem. 1969 Applications of Model Theory to Algebra, Analysis, and Probability (Internat. Sympos., Pasadena, Calif., 1967) pp. 123–137. Holt, Rinehart and Winston, New York.

[12] W.A.J. Luxemburg. *Non-Standard Analysis: Lectures on A. Robinson's Theory of Infinitesimals and Infinitely Large Numbers.* California Institute of Technology, Pasadena, California, 1973.

[15] I. Moerdijk, G.E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer, Berlin, 1991.

[20] A. Robinson. *Non-standard analysis.* North-Holland Publishing Co., Amsterdam, 1966.

[21] A. Tarski. Une contribution la thorie de la mesure, *Fund. Math.* **15** (1930), 42-50.

[22] A. Weil. *Number theory. An approach through history. From Hammurapi to Legendre.* Birkhäuser Boston, Inc., Boston, MA, 1984.

Department of Mathematics, University of Vienna, Austria.
E-mail address: paolo.giordano@univie.ac.at

Department of Mathematics, Bar Ilan University, Ramat Gan, 52900 Israel.
E-mail address: katzmik@macs.biu.ac.il